# The Asymptotic Behavior of the Stable Marriage Problem in Symmetric Markets


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## Abstract

The study of two-sided matchings—where we seek to pair agents on opposite sides of a market based on their ranked preferences—is a cornerstone of market design economics. With real-world applications ranging from school choice and medical residency to job assignment, understanding the properties of these matching markets provides immense theoretical and practical value. In this thesis, we start with the foundations of matching theory, building up a framework to analyze the set of stable matchings—ones where no matched pair mutually prefers another assignment. We then build off this work to analyze the asymptotic behavior of stable matchings, as proven by Pittel [11], [12]. Broadening our scope from dissecting individual outcomes, we instead focus on proving bounds for the expected number of stable matchings and the expected minimum and maximum rank of stable matchings in symmetric markets with complete preferences.

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# Dedication

*To my parents Jivan and Anuradha and my brother Aviyan.*

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# 1

# Foundations of Matching Markets

This chapter provides a history of two-sided matching markets, with a focus on the evolution of theoretical work and the applications that spurred them. We start with a problem that illustrates the practical importance of matching, which then motivates the foundational theory of the field. From this, we explore the connections between two-sided markets and market design more broadly, examining key applications and future directions of research. We then narrow our focus to the study of the number and rank of stable matchings, which will comprise the majority of this thesis.

## 1.1 Motivating Problem

Consider a remote village with two groups of people, whom we'll call men and women. In this village, people want to get married to someone in the opposing group and to make things easier, each person provides a ranking of potential marriage partners based on their preferences. Of course, it's possible to simply assign pairs randomly and hope for the best. However, this could lead to two quite problematic scenarios: either a person prefers to be alone than with their partner, or two people in separate marriages prefer each other to their assigned partners. These two concepts are denoted *individual rationality* and *stability* respectively and will be considered in detail in Chapter 2.

In the simplest case, if there is only one man and no women, then the matching is trivial. If we now add a woman, then there are two options: either they are married to each other in the stable matching or at least one of them prefers being alone, in which case there are no matches. Extending this line of reasoning, if there is 1 man and $n$ women, in the stable matching either the man is married to his highest ranked partner who prefers him over being alone or there are no matches.

However, what happens when we further push this reasoning to $n$ men and $n$ women? Is there always a stable matching that exists? Could there be more than one stable matching for a given setup?

## 1.2 Gale-Shapley and the Algorithmic Solution

Gale and Shapley asked, and then subsequently answered this question in their seminal 1962 paper *College admissions and the stability of marriage* [4]. In their work, the stable marriage problem is a subset of the more general college admissions framework, where $n$ applicants are applying to $m$ colleges, each of which has an integer quota $q_i \in [1, n]$ for $i \in [1, m]$. Akin to the preference orderings between men and women, each applicant and college also ranks the members of the opposing set. Gale and Shapley proved that a stable marriage always exists and provided the following important proof by construction:

**Theorem 1.1 (Deferred Acceptance Algorithm):** There always exists a set of stable marriages.

*Proof:*

**Algorithm 1** Deferred Acceptance

**Step $t = 1$:**
1. Each man proposes to his most-preferred woman
2. Each woman tentatively accepts a proposal from her highest-preference man (if any) and rejects all others.

**Steps $t \geq 2$:**
1. Each man who has been rejected in the previous step proposes to his next most-preferred woman.
2. Each woman tentatively accepts her most-preferred proposal (if any) between the one she holds and any new ones received in the round.

When there are no more rejections, the tentative proposal becomes final.

We note that this process runs for a finite number of steps—since no man can propose to the same woman more than once—meaning there are at most $(n-1)^2$ steps in the algorithm.

We see that the set of marriages found by this algorithm is always stable by contradiction: Suppose man $m_i$ and woman $w_j$ are not married to each other, but $m_i$ prefers $w_j$ to his own partner. This implies that at some step $t$, $m_i$ proposed to $w_j$, and did so before he proposed to his actual partner $w_i$. Since $w_j$ rejected $m_i$ (otherwise he wouldn't have been matched to $w_i$), she already had a preferred tentative match and her final match is at least as preferred as that tentative one, giving a contradiction. □

## 1.3 Key Applications

Matching, along with auction theory, form the backbone of *market design*, which seeks to construct solutions that increase the liquidity, efficiency, and equity in exchanges involving multiple parties [7]. Market design is broadly concerned with outcomes in *two sided* markets, which involve participants that are drawn from disjoint sets, as in our marriage example from Section 1.1. These markets come in three flavors: *one-to-one*, where each participant is matched with at most one person from the opposing party, *many-to-one*, which Gale and Shapley [4] explored with college admissions, and *many-to-many*, which we will not explore in this thesis.

Although by itself, Gale and Shapley's paper presents two seemingly stylized toy models for symmetric two-sided matching, in the decades since, a vast literature has grown out of their framework, particularly in the application of similar matching models to real-world markets. Variants of the deferred acceptance algorithm are critical in a host of diverse applications, chief among them allocating people in labor and school markets.

### 1.3.1 Medical Residency Matching

One of the earliest such applications—predating Gale and Shapley's paper—concerns the medical residency matching process. In many countries, the first step taken by doctors after completing medical school is a residency program in their specialization. Like most labor markets, participants in the matching process have their own, often conflicting, incentives: hospitals want to secure the most promising doctors for their programs while new doctors want to balance waiting for good programs with securing an offer.

Between the early 1900s and 1952, these competing goals led to two market failures. The first, called *unraveling*, occurred because hospitals competed against each other to obtain the best residents and thus each sent out offers slightly earlier than the others. This pressure moved the offer date earlier each year, until, by 1945, it was customary for medical students to be hired for residency almost two years before their graduation date [18]. The second market failure happened in 1945 when the medical schools stepped in to stop this unraveling by not releasing student information before a given date. Hospitals however, now realized that if their top choice candidates rejected their offer, their next choices would often already have accepted offers from other hospitals. In order to secure the best candidates for themselves, hospitals then started sending *exploding offers*, which had to be accepted immediately or rejected in order for a candidate to see other offers [19]. This led to a chaotic marketplace where doctors often reneged on offers and neither party reached a satisfactory matching.

These problems were solved in the early 1950s by the medical residency match, now known as the *National Resident Matching Program* (NRMP). The NRMP is a centralized clearinghouse that consolidates the preference lists of doctors and hospitals. It then runs a *hospital-proposing* deferred acceptance algorithm over the orderings to produce a stable matching [14]. Since then, the NRMP has been further modified in order to overcome new hurdles, chief among them the desire to keep couples—of which there were many—near each other.

### 1.3.2 School Choice

Another problem whose basic form presents itself as a stable matching application is the assignment of students to schools—something that more closely mirrors the college admissions model of Gale and Shapley [4]. The deferred acceptance algorithm has been used for specialized high school admissions in New York City schools as well as for general admissions in Boston public schools [1], the former of which is a two-sided market where students and schools rank each other while the latter is a one-sided market where schools do not rank applicants. In these instances, the original matching algorithms employed through a centralized clearinghouse were not strategy proof, meaning families could benefit by gaming the system by submitting preference lists that weren't accurate to their true desires [2]. This resulted in a market failure different from the medical residency match, where instead of an unraveling or a chaotic marketplace, participants—usually the families with more resources and institutional know-how—could improve their chances of being placed in a preferred school [15]. In these instances, deferred acceptance algorithms, which are strategy proof for the proposing side, thus enabled participants to state their true preferences and reduced the impact of outside factors in school placement.

## 1.4 Scope of the Thesis

A key finding of Gale and Shapley's work was the *existence* of a stable matching [4]. However, a market can have multiple possible assignments that all satisfy stability conditions, each with different properties with regards to the outcomes. In this expository thesis, we thus study the theory behind determining the *number* of stable matchings in an instance of the stable marriage problem. We tackle this problem from the lens of Pittel [11] by extending the work of Knuth [6] to first compute the expected number of stable matchings in a symmetric market with $n$ participants on each side and then determine asymptotic bounds for the number of stable matchings as $n \to \infty$ for preferences chosen uniformly at random [12]. These results are then used to provide bounds for the expected maximum and minimum ranks in a stable marriage, which are indicators for the relative 'desirability' of a matching for the set of men or women as a whole.

# 2

# Mathematics of Stable Matching

This chapter is an introduction to the formal mathematics of matching theory. We start with the definitions and theorems that lay the groundwork for the marriage model discussed in Chapter 1. This is followed by an exploration of stability in the marriage problem and the class of stable outcomes. Key properties of stable matchings are then proven, followed by an introduction to the lattice framework for the set of stable matchings. The direction for many of the definitions and theorems are adapted from Roth and Sotomayor [17].

## 2.1 The Marriage Model

The players in our game are a part of two finite, disjoint sets $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_p\}$, denoted men and women respectively. Together, they form a two-sided market, meaning each man has preferences over himself (i.e. remaining unmatched) and the set of women, and each woman has preferences over herself and the set of men. For a man $m_i$ and women $w_j, w_k$ we write $w_j \succ_{m_i} w_k$ to denote that $m_i$ prefers $w_j$ over $w_k$ and we write $w_j \succeq_{m_i} w_k$ to denote that $m_i$ prefers $w_j$ at least as much as $w_k$.

Formally, we adapt the notation of Roth and Sotomayor [17] as follows: for a man $m_i \in M$, his preferences are denoted by an ordered list $P(m_i)$ over the set $\{m_i\} \cup W$. Consider as an example the following preference:

$$P(m_i) = w_j, w_k, m_i, \ldots, w_l \tag{2.1}$$

This indicates that $w_j \succ_{m_i} w_k$. Any woman $w$ such that $w \succeq_{m_i} m_i$ is called *acceptable* and for brevity, we will only write preference lists up to acceptability. The above example then becomes

$$P(m_i) = w_j, w_k \tag{2.2}$$

Finally, it's possible for a man $m_i$ to be *indifferent* between some set of women $\{w_j, w_k\} \subseteq W$. Indifference between these alternatives is denoted $[w_j, w_k]$.

**Definition 2.1 (Strict Preferences):** If an individual is not indifferent between any alternatives, their preferences are said to be strict. Unless stated otherwise, our markets in this work will involve strict preferences.

The set of all preference lists is denoted $\mathbf{P} = \{P(m_1), \ldots, P(m_n), P(w_1), \ldots, P(w_p)\}$. A marriage market can then be defined completely by the tuple $(M, W, \mathbf{P})$. We will make one more assumption about the structure of preferences:

**Definition 2.2 (Rationality):** Individuals are said to be rational if their preferences form a *complete ordering* and are *transitive*.

In this context, a complete ordering means that for any $m_i$, over the set of alternatives $W$, any two alternatives can be compared to each other. Transitivity, on the other hand, means that for alternatives $w_j, w_k, w_l \in W$, if $w_j \succ_{m_i} w_k$ and $w_k \succ_{m_i} w_l$, then $w_j \succ_{m_i} w_l$. Because our market is two-sided, all of these concepts apply symmetrically to women as well.

The output of a marriage market, such as the one constructed above, is a set of marriages, called a matching.

**Definition 2.3 (Matching):** A matching is a function $\mu \colon M \cup W \to M \cup W$ such that $\mu(m) \in W \cup \{m\}, \mu(w) \in M \cup \{w\}$ and $u^2(m) = m, u^2(w) = w$.

Intuitively, the first condition means that the market is two-sided and that each person is either matched with themselves or with someone from the opposing set. The second condition implies that if $m$ is matched to $w$ then $w$ is matched to $m$, since $\mu(\mu(m)) = \mu(w) = m$ and vice versa.

## 2.2 Stability in the Marriage Model

Going beyond just describing matchings, we will now delve into why some outcomes are more likely to occur than others in our game. To do so, we first extend **Definition 2.2** to describe matchings as follows:

**Definition 2.4 (Individual Rationality):** A matching $\mu$ is said to be individually rational if all agents $i$ find their partners $\mu(i)$ to be acceptable.

This means that there is no individual $k$ in our matching who prefers themselves to their given match: $k \not\succ_k \mu(k)$. In addition to a match not being blocked by an individual, to ensure that there are no "divorces," we also want there to be no pairs who are incentivized to go against the outcome.

**Definition 2.5 (Blocking Pair):** A blocking pair to a matching $\mu$ is a pair of agents $(m, w)$ who prefer each other to their matches under $\mu$, meaning $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

In this scenario, the pair $(m, w)$ are incentivized to ignore $\mu$ and marry each other, meaning they block the matching.

**Definition 2.6 (Stable Matching):** A matching $\mu$ is stable if it is individually rational and contains no blocking pairs.

## 2.3 Key Properties of Stable Marriages

Having established the formal definition of a stable matching, we now refer back to Chapter 1, where we showed that the deferred acceptance algorithm [4] was finite and produced a stable marriage. There, we noted that the algorithm found *a* stable matching out of many that may be possible for a given setup. We now categorize these types of matchings. Let $\succ_M$ be a partial ordering on the set of stable matches, which represents the common preferences of all men. Then, $\mu \succ_M \mu'$ means that all men like $\mu$ at least as much as $\mu'$ and there exists at least one man for which this preference is strict. $\succ_W$ is defined analogously for women.

**Definition 2.7 (M/W-Optimal Stable Matching):** Given a market $(M, W, \mathbf{P})$, a stable matching $\mu$ is Man-optimal (M-optimal) if for all other stable matchings $\mu'$, $\mu \succeq_M \mu'$. Similarly, $\mu$ is Woman-optimal (W-optimal) if $\mu \succeq_W \mu'$.

We can now show that the man-proposing deferred acceptance algorithm returns the $M-$optimal stable matching (and likewise for the woman-proposing variant). To do so, we define the notion of a pair $(m, w)$ being *achievable* for each other if in a market $(M, W, \mathbf{P})$ there exists some $\mu$ such that $m = \mu(w)$ and vice versa.

**Theorem 2.8 (Gale and Shapley [4]):** When all men and women have strict preferences, the man-proposing deferred acceptance algorithm produces the M-optimal stable matching.

*Proof:* Assume that up to some step $t'$ in the deferred acceptance algorithm, no man has been rejected by an achievable woman. Now, suppose there exists a pair $(m, w)$ where $w$ rejects $m$ on step $t'$. We have two cases:

- **Case I:** $w$ finds $m$ unacceptable, in which case $w$ is unachievable for $m$, which is fine.
- **Case II:** $w$ rejects $m$ in favor of $m'$ who she prefers more. In this case, we know that $m'$ prefers $w$ to all women other than those he has already been rejected by up to step

$t'$. Let $\mu$ be a matching that pairs $(m, w)$ and everyone else to someone achievable. Then, we know that $w \succ_{m'} \mu(m')$ and from earlier we know that $m' \succ_w m$, meaning $(m', w)$ form a blocking pair for any such $\mu$, meaning $w$ is always unachievable for $m$.

Thus, there is no stable matching that pairs $m, w$, as desired. □

From this, we thus see that in the setting of strict preferences, each side of the market agrees on its own best stable matching—the men prefer $\mu_M$ produced by the man-proposing DA algorithm and the women prefer $\mu_W$ likewise. These preferences are not only distinct, but also *opposing*: the optimal stable matching for men turns out to be the *worst* stable matching for women and vice versa.

**Theorem 2.9 (Knuth [6]):** When all agents have strict preferences, for stable matchings $\mu, \mu'$ we have $\mu \succ_M \mu'$ if and only if $\mu' \succ_W \mu$.

*Proof:* Let $\mu, \mu'$ be stable matchings such that $\mu \succ_M \mu'$ and assume that $\mu' \not\succ_W \mu$. We proceed by contradiction. This implies that $\exists w \in W$ such that $\mu \succ_w \mu'$. Because $w$ prefers $\mu$ to $\mu'$, she must have different partners in the two matchings. Moreover, individual rationality implies that she can't be matched to herself in $\mu$ so we can let $m = \mu(w)$. Because $\mu \succ_M \mu'$, we have that $w \succ_m \mu'(m)$, meaning that $(m, w)$ form a blocking pair for $\mu'$ and thus $\mu'$ is not stable, a contradiction. □

For a given market $(M, W, \mathbf{P})$ and stable matchings $\mu, \mu'$ we further define $M(\mu), W(\mu)$ to be the set of men, women who prefer $\mu$ to $\mu'$ and similarly $M(\mu'), W(\mu')$ to be the set of men, women who prefer $\mu'$ to $\mu$. Knuth [6] then proved the following about these sets:

**Theorem 2.10 (The Decomposition Lemma):** For stable matchings $\mu, \mu'$ over $(M, W, \mathbf{P})$

where all preferences in $\mathbf{P}$ are strict, we have that the maps $\mu, \mu'$ from $M(\mu) \to W(\mu')$ and $M(\mu') \to W(\mu)$ are surjective.

*Proof:* Choose $m \in M(\mu)$, which means $\mu(m) \succ_m \mu'(m)$ and note by definition $\mu'(m) \succeq_m m$, meaning $\mu(m) \in W$. Let $w = \mu(m)$. Then, by stability of $\mu'$ we must have that $\mu'(w) \succ_w \mu(w) = m$. Thus, we have that $w \in W(\mu')$ and thus $\mu(M(\mu)) \subseteq W(\mu')$. By a symmetric argument, $\mu'(W(\mu')) \subseteq M(\mu)$. Therefore, because $M(\mu), W(\mu')$ are necessarily finite and $\mu, \mu'$ are one-to-one, they must also be surjective. □

The decomposition lemma allows us to prove many non-obvious facts about stable matchings. In particular, McVitie and Wilson [8] showed the following about the set of people who are single (and thus the set who are partnered) in all stable matchings:

**Theorem 2.11 (The Lone Wolf Theorem):** For a given market $(M, W, \mathbf{P})$ where the preferences are strict, the set of people who are single (partnered) is identical over all stable matchings.

*Proof:* We will show this by contradiciton. Suppose, for stable matches $\mu, \mu'$ there exists $m \in M$ such that $m$ has a match under $\mu'$ but not under $\mu$. This means $m \in M(\mu')$. Applying Theorem 2.10, we know that $\mu\colon W(\mu) \to M(\mu')$ is surjective, meaning $m$ must also be matched under $\mu$, which is a contradiction. □

In our study of matching in two-sided markets, we've made the implicit assumption that the most desirable outcome is a stable one. Looking at the $M$–optimal stable match, it is clear by definition that it is preferred to all other stable matches for the men. However, we haven't yet shown that the $M$–optimal stable match is preferred by the men over any *unstable* match. After all, if such a match existed, it would imply that optimizing for stability comes at some cost to the total utility of all men. As always, all concepts apply symmetrically to $W$–optimal stable matches. One way to compare different stable matchings and their aggregate utility to the participants is through the notion of Pareto optimality [10].

**Definition 2.12 (Pareto Optimality for Men):** Consider a market $(M, W, \mathbf{P})$ and a matching $\mu : m \to \mu(m) \in W$. $\mu$ is Pareto optimal if there exists no other matching $\mu'$ such that $\mu'(m) \succ_m \mu(m)$ for *some* $m \in M$.

This is quite a strong construct: if a situation is Pareto optimal, then the situation can't be strictly improved for *any* man without harming the outcomes of other men. We can loosen this definition a bit by looking at the space of matchings that can't be improved upon for *every* individual involved.

**Definition 2.13 (Weak Pareto Optimality):** Consider a market $(M, W, \mathbf{P})$ and a matching $\mu : m \to \mu(m) \in W$. $\mu$ is weak Pareto optimal if there exists no other matching $\mu'$ such that $\mu'(m) \succ_m \mu(m)$ for *all* $m \in M$.

Having established these definitions, we will now show a perhaps unintuitive fact: that no matching, regardless of stability, is preferred by all men to the $M$–optimal stable match.

**Theorem 2.14 (Weak Pareto Optimality for Men):** The $M$–optimal stable match $\mu_M$ is weak Pareto optimal for the set of men, meaning there exists no matching $\mu$ such that $\mu_M \succ_m \mu$ for all $m \in M$.

*Proof:* We will show this by contradiction. Let $\mu$ be such a matching and note that by definition of $\mu$, $\forall m \in M$, $\mu(m) = w \in W$ such that $\mu_M(w) = m' \neq m$, meaning $\mu$ matches every man to an acceptable woman who rejected him under $\mu_M$. Thus, everyone matched under

$\mu$ is also matched under $\mu_M$, meaning $\mu_M(\mu(M)) = M$. This implies $\mu_M(M) = \mu(M)$.

Now consider the man-proposing deferred acceptance algorithm used to generate $\mu_M$. Note that $\mu_M$ matches all men in $M$ and the algorithm halts as soon as every woman in $\mu_M(M)$ has an acceptable proposal. Consider a woman $w \in W$ who gets a proposal in the final proposing step of the algorithm. She hasn't rejected any acceptable men yet by construction. However, because we assumed that $\mu \succ_m \mu_M$ for all $m \in M$, we must have that this woman is unmatched in $\mu_M$. This contradicts the fact that $\mu_M(M) = \mu(M)$. $\square$

## 2.4 The Lattice Structure of Stable Outcomes

Working in the framework of strict preferences, for arbitrary matchings $\mu, \mu'$ define the following functions on the set $M \cup W$:

$$\lambda(m) = \mu \vee_M \mu' = \begin{cases} \mu(m) \text{ if } \mu(m) \succ_m \mu'(m) \\ \mu'(m) \text{ else} \end{cases} \tag{2.3}$$

$$\lambda(w) = \mu \vee_M \mu' = \begin{cases} \mu(w) \text{ if } \mu'(w) \succ_w \mu(w) \\ \mu'(w) \text{ else} \end{cases} \tag{2.4}$$

The functions $\nu(m) = \mu \wedge_M \mu'$ and $\nu(w) = \mu \wedge_M \mu'$ are defined analogously, with the preferences reversed. $\lambda$ is thus the function that assigns each man his more preferred partner between $\mu, \mu'$ and each woman her less preferred partner while $\nu$ is the function that assigns each man his less preferred partner and each woman her more preferred one.

While these are interesting constructions, it's not immediately clear whether $\lambda, \nu$ are themselves matchings. We could have that for $m \neq m' \in M$, $\lambda(m) = \lambda(m')$, meaning the same woman is the most preferred for two different men. Additionally, it may be that $\lambda(m) = w$ but $\lambda(w) \neq m$ for some pair $(m, w)$, meaning the outputs of the function aren't opposing for men and women. None of these turn out to be the case, however, as shown by John Conway in [6].

**Theorem 2.15 (Lattice Theorem):** Under strict preferences, given stable matchings $\mu, \mu'$ the functions $\lambda = \mu \vee_M \mu'$ and $\nu = \mu \wedge_M \mu'$ are also stable matchings.

*Proof:* We show this for $\lambda$ and note that a symmetric argument applies to $\nu$. First, to prove that $\lambda$ is a matching at all, we must show that $\lambda(m) = w \iff \lambda(w) = m$. The direction $\lambda(m) = w \Longrightarrow \lambda(w) = m$ follows from stability of $\mu, \mu'$.

For the other direction, construct the sets $M^* = \{m \in M | \lambda(m) \in W\}$ and $W^* = \{w \in W | \lambda(w) \in M\}$. By the definition of $\lambda$, $M^* = \{m \in M | \mu(m)$ or $\mu'(m) \in W\}$ and $W^* = \{w \in W | \mu(w)$ or $\mu'(w) \in M\}$. Now, we utilize the other direction, which tells us that $\lambda(M^*) \subseteq W^*$. Now, note that $|\lambda(M^*)| = |M^*|$ since $\lambda(m) = \lambda(m^*) = w$ if $m = m^* = \lambda(w)$. $\lambda(M^*)$ is at least as large as $\mu(W^*)$, which is the same size as $W^*$, meaning $\lambda(M^*)$ and $W^*$ are of the same size and $\lambda(M^*) = W^*$. We conclude by considering the following two cases:

- **Case I:** If $w \in W*$ then for some $m \in M$, $\lambda(w) = m$ and thus $\lambda(m) = w$.
- **Case II:** If $w \notin W*$ then $\lambda(w) = w$. Thus, if $\lambda(w) = m$ then $\lambda(m) = w$.

Finally, we show that $\lambda$ is stable by contradiction. Suppose there exists a blocking pair $(m, w)$ for $\lambda$. This implies $w \succ_m \lambda(m)$, meaning $w \succ_m \mu(m)$ and $w \succ_m \mu'(m)$. We also have $m \succ_w \lambda(w)$, meaning we must either have that $\lambda(w) = \mu(w)$ or $\lambda(w) = \mu'(w)$. In either case, this contradicts the stability of $\mu, \mu'$. $\square$

We can use Conway's result to study the macro structure of the set of stable matchings.

**Definition 2.16 (Lattice):** A lattice is any set $L$ with a partial ordering $\succeq$ for which all pairs of elements $a, b \in L$ have a supremum $a \vee b \in L$ and infimum $a \wedge b \in L$.

**Definition 2.17 (Distributive Lattice):** A lattice $L$ is distributive if all of its elements satisfy the distributive property: for any $a, b, c \in L$ we have

$$a \vee (b \vee c) = (a \vee b) \vee c$$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

**Theorem 2.17:** The set of all stable matchings is a distributive lattice.

*Proof:* Let $\mu, \mu', \mu''$ be three arbitrary stable matchings. First, note by our definitions of $\vee, \wedge$ that both functions are symmetric about their inputs, meaning they are commutative. From Theorem 2.15, we note that applying $\vee$ or $\wedge$ to any pair also results in a stable matching. The distributive property then follows. $\square$

# 3

# The Number of Stable Matchings

Our analysis thus far has revolved around specific, special stable matchings in the two-sided marriage market. In this chapter, we zoom out in scope, building on the theory introduced in Chapter 2 to explore the expected number of stable matchings in a size $n$ stable marriage problem with strict preferences chosen uniformly and at random. This work expands and adds more detail to the theorems of Pittel [11], particularly as it pertains to random partitions of the unit interval.

## 3.1 A Loose Lower Bound

Given a particular market $(M, W, \mathbf{P})$, one way to understand the space of stable matchings is to algorithmically find each and every one. As we'll see, however, the number of stable matchings can grow exponentially in the size of the market, making this infeasible in many scenarios. We thus start by providing a lower bound on the number of stable matchings.

Specifically, for a market of size $n = |M| = |W|$, denote by $f(n)$ the number of stable matchings and assume that all preference lists have self-matches as the least desired outcome. We start by showing that $f(n)$ indeed grows exponentially in $n$.

**Theorem 3.1:** Given two marriage markets $(M, W, \mathbf{P})$ and $(M^*, W^*, \mathbf{P}^*)$ of size $a, b$ with $x, y$ stable matchings respectively, there exists a market of size $ab$ with at least $\max(xy^a, yx^b)$ stable matchings.

*Proof:* Label the set of men and women in the two markets by $(m_i, w_i), (m_i^*, w_i^*)$. We now construct a market of size $ab$ as follows:

- Each man is denoted by the tuple $(m_i, m_j^*)$ and each woman by the tuple $(w_i, w_j^*)$ where $i \in [1, a], j \in [1, b]$.
- $(w_k, w_l^*) \succ_{(m_i, m_j^*)} (w_{k^*}, w_{l^*}^*)$ if $w_l^* \succ_{m_j^*} w_{l^*}^*$ or if $l = l^*$ and $w_k \succ_{m_i} w_{k^*}$.
- $(m_k, m_l^*) \succ_{(w_i, w_j^*)} (m_{k^*}, m_{l^*}^*)$ if $m_l^* \succ_{w_j^*} m_{l^*}^*$ or if $l = l^*$ and $m_k \succ_{w_i} m_{k^*}$.

Now, let $\mu_1, \ldots, \mu_b$ be any set of stable matchings for market $(M, W, \mathbf{P})$ and $\mu$ be any stable matching for market $(M^*, W^*, \mathbf{P}^*)$. We will show that the following map is a stable matching: $\gamma_j \colon (m_i, m_j^*) \to (\mu_j(m_i), \mu(m_j^*))$.

First, note that $\gamma_i$ is a matching, since both $\mu, \mu_i$ are, and thus the properties of **Definition 2.3** are satisfied. We now proceed by contradiction. Assume that $\gamma_i$ is blocked by a pair $((m, m^*), (w, w^*))$. We then have four possibilities for preferences between the alternatives:

- **Case I:** $w^* \succ_{m^*} \mu(m^*)$ and $m^* \succ_{w^*} \mu(w^*)$. This scenario is impossible because it violates stability of $\mu$.
- **Case II:** $w^* \succ_{m^*} \mu(m^*)$ and $m^* = \mu(w^*)$, $m \succ_w \mu_j(m)$. Conditions are incompatible.
- **Case III:** $m^* \succ_{w^*} \mu(w^*)$ and $w^* = \mu(m^*)$, $w \succ_m \mu_j(w)$. This scenario is impossible because it violates stability of $\mu_j$.
- **Case IV:** $m^* \succ_{w^*} \mu(w^*)$ and $w^* \succ_{m^*} \mu(w^*)$. Conditions are incompatible.

Thus, the map $\gamma_i$ is stable and there are $yx^b$ such stable matchings. A symmetric argument shows the corresponding result for $xy^a$. □

Having established the exponential growth of $f(n)$, we now show a lower bound on the number of stable matches.

**Theorem 3.2:** For $k \in \mathbb{Z}^+$, there exists a marriage market $(M, W, \mathbf{P})$ of size $|M| = |W| = n = 2^k$ with at least $2^{n-1}$ stable matchings.

*Proof:* We proceed by induction.

- *Base Case:* When $n = 2^0$, the market has 1 participant on either side, thus admitting the trivial stable marriage. For $n = 2^1$, the deferred acceptance algorithm provides one stable marriage.

- *Inductive Hypothesis:* Assume the theorem holds for $n = 2^k$.
- Let the stable matching from the hypothesis have size $b = n$ with $y = 2^{2^k-1}$ and construct a stable matching of size $a = 2$ with $x = 2$ stable matches as follows:

$$w_1 \succ_{m_1} w_2, w_2 \succ_{m_2} w_1 \tag{3.1}$$

$$m_2 \succ_{w_1} m_1, m_1 \succ_{w_2} m_2 \tag{3.2}$$

We can now apply **Theorem 3.1**, which tells us that there exists a marriage market of size $2 \cdot 2^{2^k} = 2^{k+1}$ with at least

$$\max\left(2 \cdot \left(2^{2^k-1}\right)^2, 2^{2^k} \cdot 2^{2^k-1}\right) = 2^{2^{k+1}-1} \tag{3.3}$$

stable matchings, as desired. □

## 3.2 Probability a Matching is Stable

In order to improve on our results from the previous section and provide a tighter asymptotic bound on the expected number of stable matchings, we first derive expressions for the probability of a given match being stable, providing a formal proof for the claims made by Knuth [6].

We work in the space of marriage markets $(M, W, \mathbf{P})$ of size $n$, meaning $|M| = |W| = n$ where all preference lists are strictly ordered and contain every member of the opposing set—meaning being single is the least-preferred outcome for each participant. Each person has $n!$ distinct choices for their preference list, meaning there are a total of $(n!)^{2n}$ possible configurations for such a market.

Let every participant choose their preference lists at random and note that by symmetry, each possible matching has the same probability $P_n$ of being stable. Construct the $n \times n$ square matrix $\mathbf{X} = [x_{ij}]$ with diagonal elements $x_{ii} = 0 \ \forall i \in [n]$ and with $x_{ij} = 0$ or $1 \ \forall i \neq j \in [n]$. Note that there are $2^{n(n-1)}$ such matrices. In addition, define the row and column sums of $\mathbf{X}$ as $r_i = \sum_j x_{ij}$ and $c_j = \sum_i x_{ij}$.

**Theorem 3.3 (Knuth [6]):** The probability of a random matching $\mu$ being stable is

$$P_n = \sum_{[x_{ij}]} \prod_{1 \leq j \leq n} \frac{(-1)^{r_j}}{(1 + r_j)(1 + c_j)} \tag{3.4}$$

*Proof:* For any given matrix $\mathbf{X}$, define $B_{\mathbf{X}}$ to be the event that the man-woman pair $(i, j)$ corresponding to entries where $x_{ij} = 1$ blocks $\mu$. Thus, if $\mathbf{X} = 0$, then there are no blocking pairs and we have $r_j = c_j = 0 \ \forall j$, meaning the above formula is valid, since the match is stable:

$$P_n = \sum_{[x_{ij}]} \prod_{1 \leq j \leq n} \frac{(-1)^0}{(1 + 0)(1 + 0)} = \sum_{[x_{ij}]} \prod_{1 \leq j \leq n} 1 = 1 \tag{3.5}$$

We now proceed to calculate $P_n$ using inclusion-exclusion, subtracting the number of matches that are blocked. To calculate the number of matches that are blocked by a given event $B_{\mathbf{X}}$, note that for a given row $i \in [n]$ there are $r_i$ women that block $\mu$, meaning each of these women must be before $w_i$ on the preference list of $m_i$ and for a given column $j \in [n]$ there are $c_j$ men that block $\mu$, meaning each of these men must be before $m_j$ on the preference list of $w_j$.

For each $j \in [n]$ we thus have that the number of blocked stable marriage instances is

$$\frac{(n!)^2}{(1+r_j)(1+c_j)} \tag{3.6}$$

Taking the product over all rows, the total number of blocked marriages due to $B_{\mathbf{X}}$ is

$$\prod_{1\leq j\leq n} \frac{(n!)^2}{(1+r_j)(1+c_j)} = (n!)^{2n} \prod_{1\leq j\leq n} \frac{1}{(1+r_j)(1+c_j)} \tag{3.7}$$

Since there are $(n!)^{2n}$ possible market configurations, we have

$$P(B_{\mathbf{X}}) = \prod_{1\leq j\leq n} \frac{1}{(1+r_j)(1+c_j)} \tag{3.8}$$

The number of nonzero entries in a given matrix $\mathbf{X}$ is equal to the sum over all rows of $r_j$. Our desired probability is then, by inclusion-exclusion,

$$P_n = 1 + \sum_{\mathbf{X}\neq 0} (-1)^{r_j} P(B_{\mathbf{X}}) \tag{3.9}$$

Combining with equation 3.5 we have our desired probability

$$P_n = \sum_{[x_{ij}]} \prod_{1\leq j\leq n} \frac{(-1)^{r_j}}{(1+r_j)(1+c_j)} \tag{3.10}$$

□

We can now use this to derive the following integral formula for the probability of a stable match:

**Theorem 3.4 (Knuth [6]):** The expression from **Theorem 3.3** is equivalent to

$$P_n = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \prod_{1 \le i \ne j \le n} (1 - x_i y_j) dx_1 \dots dx_n dy_1 \dots dy_n \text{ where } x_i, y_j \in [0,1] \tag{3.11}$$

*Proof:* We can rewrite the above product as

$$\prod_{1 \le i \ne j \le n} (1 - x_i y_j) = \sum_{[x_{ij}]} \prod_{1 \le i,j \le n} (-x_i y_j)^{[x_{ij}]} \tag{3.12}$$

$$= \sum_{[x_{ij}]} \prod_{1 \le j \le n} (-x_j)^{r_j} y_j^{c_j} \tag{3.13}$$

We can now take $2n$ integrals of the sum over $[0,1]$, which gives our desired alternating sum from equation 3.10.

## 3.3 Core Theorems

Before utilizing our work from the previous section to formulate a tighter bound on the number of stable matchings, we first explore foundational definitions and prove well-known bounds on random variables which will be useful in the constructions to come.

### 3.3.1 Random Variables Over the Unit Interval

Let $U_1, \ldots, U_n$ be independent, identically distributed $U[0,1]$ random variables and define the functions:

$$S_n = \sum_{j=1}^{n} U_j \qquad\qquad T_n = \frac{\sum_{j=1}^{n} U_j^2}{S_n^2} \tag{3.14}$$

In addition, we construct a random partition of $[0,1]$ as follows: select the $n-1$ random points over $[0,1]$ as the values of the random variables $U_1, \ldots, U_{n-1}$. Consider the order statistics over this distribution, which gives us a reordering $U_{(1)}, \ldots, U_{(n-1)}$ where $U_{(i)} \leq U_{(i+1)}$. We can now partition the unit interval into disjoint sub-intervals

$$L_1 = [0, U_{(1)}), L_2 = [U_{(1)}, U_{(2)}), \ldots, L_n = [U_{(n-1)}, 1] \tag{3.15}$$

Note that $\sum_i^n L_i = 1$, meaning $L_1, \ldots, L_{n-1}$ jointly determine $L_n$, and define the quantities

$$V_n = \sum_{j=1}^{n} L_j^2 \qquad\qquad M_n = \max_{1 \leq j \leq n} L_j \tag{3.16}$$

**Theorem 3.5:** The joint density of $(L_1, \ldots, L_{n-1})$ equals $(n-1)!$ whenever it is not $0$.

*Proof:* Note that the sample space corresponding to $(U_1, \ldots, U_{n-1})$ is $[0,1]^{n-1}$ and let our measure over the hypercube be the Lebesgue measure. We can then define $\Gamma := \{0 \leq u_1 \leq \cdots \leq u_{n-1} \leq 1\}$ as the subset of the hypercube corresponding to $(U_{(1)} \leq \cdots \leq U_{(n-1)})$. There are $(n-1)!$ permutations $\omega$ of $[1, n-1]$ and thus $[0,1]^{n-1}$ contains $(n-1)!$ such regions $\Gamma_\omega := \{0 \leq u_{\omega(1)} \leq \cdots \leq u_{\omega(n-1)} \leq 1\}$. Let $M \subseteq \Gamma$ be a measurable subset with corresponding $M_\omega \subseteq \Gamma_\omega$. We then have:

$$P((U_{(1)} \leq \cdots \leq U_{(n-1)}) \in M) = P((U_1, \ldots, U_{n-1}) \in \cup_\omega M_\omega) \tag{3.17}$$

$$= \sum_\omega P((U_1, \ldots, U_{n-1}) \in M_\omega) \tag{3.18}$$

$$= (n-1)! P((U_1, \ldots, U_{n-1}) \in M) \tag{3.19}$$

Where the final equality comes from the fact that each $U_i$ is a $U[0,1]$ random variable. We thus see that the measure of $(U_{(1)} \leq \cdots \leq U_{(n-1)})$ is $(n-1)!$ over every measurable subset of $\Gamma$ and $0$ otherwise.

To translate this result into one over $(L_1, \ldots, L_{n-1})$, let $U_{(0)} := 0$ and define

$$L_i := U_{(i)} - U_{(i-1)} \qquad L_n := 1 - \sum_{i=1}^{n-1} L_i \tag{3.20}$$

The transformation $\alpha\colon (U_{(1)} \leq \cdots \leq U_{(n-1)}) \to (L_1, \ldots, L_{n-1})$ has a Jacobian matrix whose determinant has absolute value one, so we have that $\alpha$ maps

$$\Gamma \to \Omega = \{(\ell_1, \ldots, \ell_{n-1}) \geq \vec{0}\colon \sum_i \ell_i \leq 1\} \tag{3.21}$$

Thus, $(L_1, \ldots, L_{n-1})$ has density $(n-1)!$ over all $(\ell_1, \ldots, \ell_{n-1}) \in \Omega$ and zero otherwise. □

### 3.3.2 Functions of Subintervals Over [0,1]

We now prove three lemmas on the likely asymptotic behavior of functions of these subintervals that will be critical in the main bound of Pittel [11].

**Lemma 3.6:** For $M_n$ as defined above and $\epsilon > 0$

$$\lim_{n\to\infty} P_n := P\left(M_n \geq \frac{(1+\epsilon)\ln(n)}{n}\right) = 0 \tag{3.22}$$

*Proof:* Because we showed above that the density of $(L_1, \ldots, L_{n-1})$ is uniform over $\Omega$, note that $P(L_1 \geq x) = P(L_i \geq x)$, meaning we can union bound:

$$P_n \leq nP\left(L_1 \geq \frac{(1+\epsilon)\ln(n)}{n}\right) \tag{3.23}$$

However, the right hand side of our expression is equivalent to

$$nP\left(L_1 \geq \frac{(1+\epsilon)\ln(n)}{n}\right) = nP\left(U_{(1)} = \min_{1\leq i\leq n} U_{(i)} \geq \frac{(1+\epsilon)\ln(n)}{n}\right) \tag{3.24}$$

$$= n\left(1 - \frac{(1+\epsilon)\ln(n)}{n}\right)^n \tag{3.25}$$

The limit of which goes to 0 as $n \to \infty$, as desired. □

Consider two distinct partitions of the unit interval $[0,1]$, which we will denote $\vec{L} = (L_1, \ldots, L_n)$ and $\vec{K} = (K_1, \ldots, K_m)$. For $m = n + o(n)$, the following limit holds:

**Lemma 3.7:** $\lim_{m\to\infty}\left(m\sum_{i\in[m]} L_i K_i\right) = 1$.

*Proof:* For random variables $X_i$, $Y_j$ which are independent and exponentially distributed with parameter 1, the distribution $(L_1, \ldots, L_{m-1})$ is equivalent to the distribution

$$\left( \frac{X_1}{\sum_{i \in [m]} X_i}, \ldots, \frac{X_{m-1}}{\sum_{i \in [m]} X_i} \right) \tag{3.26}$$

And the distribution $(K_1, \ldots, K_{n-1})$ is equivalent to

$$\left( \frac{Y_1}{\sum_{j \in [n]} Y_j}, \ldots, \frac{Y_{n-1}}{\sum_{j \in [n]} Y_j} \right) \tag{3.27}$$

We can thus write our expression as

$$m \sum_{i \in [m]} L_i K_i = \frac{\frac{1}{m} \sum_{i \in [m]} X_i Y_i}{\frac{1}{m} \sum_{i \in [m]} X_i \cdot \frac{1}{n} \sum_{j \in [n]} Y_j} \left( \frac{m}{n} \right) \tag{3.28}$$

We can now use the weak law of large numbers, which tells us that the following expressions converge in probability:

$$\frac{1}{m} \sum_{i \in [m]} X_i Y_i \to 1 \qquad \frac{1}{m} \sum_{i \in [m]} X_i \to 1 \qquad \frac{1}{n} \sum_{j \in [n]} Y_j \to 1 \tag{3.29}$$

We combine these with the observation that because $m = n + o(n)$, $\lim_{n \to \infty} \frac{m}{n} = 1$ to get our desired limit. □

**Lemma 3.8:** Using the definitions at the beginning of Section 3.3.1, we have

$$\lim_{m \to \infty} m V_m = 2 \tag{3.30}$$

*Proof:* We use the same definiton of $X_i$ as in **Lemma 3.7** and simplify the expression

$$mV_m = \frac{m\sum_{i=1}^m X_i^2}{(\sum_{i=1}^m X_i)^2} = \frac{m^{-1}\sum_{i=1}^m X_i^2}{(m^{-1}\sum_{i=1}^m X_i)^2} \to \frac{E[X_i^2]}{E[X_i]^2} = 2 \tag{3.31}$$

Where the final equality comes from the expectation and variance of the exponential distribution. □

We can now use the above lemmas in proving the following bound, which will be crucial to our eventual result:

**Theorem 3.9:** Let $S_k$ be the sum of $k$ $U[0,1]$ random variables as described above and $f_{S_k}(s)$ the density of $S_k$. Then,

$$f_{S_k}(s) = \frac{s^{k-1}}{(k-1)!}P(M_k \leq s^{-1}) \leq \frac{s^{k-1}}{(k-1)!} \tag{3.32}$$

*Proof:* For $0 < s_1 < s_2 < n$ we can write the probability of the event $S_k \in [s_1, s_2]$ as an integral over a subset of $[0,1]^n$ as

$$P(s_1 \leq S_k \leq s_2) = \underbrace{\int \cdots \int}_{s_k \in [s_1, s_2]} du_1 \dots du_k \tag{3.33}$$

Where $u_i \in [0,1]$ and $s_k = \sum_i^k u_i$. We now transform into new variables by the map

$$\alpha\colon (u_1, \dots, u_k) \to (s, \ell_1, \dots, \ell_{k-1}) \text{ where } \ell_i = \frac{u_i}{s} \tag{3.34}$$

Defining $\ell_k = 1 - \sum_i^{k-1} \ell_i$, we have the inverse transformation $\alpha^{-1}\colon u_i \to s\ell_i$ with Jacobian

$s^{k-1}$. We can now write the probability as

$$P(s_1 \leq S_k \leq s_2) = \int \cdots \int s^{k-1} ds d\ell_1 \dots d\ell_{k-1} \tag{3.35}$$

$$= \int_{s_1}^{s_2} \frac{s^{k-1}}{(k-1)!} \underbrace{\int \cdots \int}_{\ell_i \leq s^{-1}} (k-1)! d\ell_1 \dots d\ell_{k-1} \tag{3.36}$$

Letting $L_i$ once again be defined as the $i^{th}$ subinterval in a random partition of $[0, 1]$ by choosing $k-1$ points from $U[0, 1]$, we use our proof of the fact that $(L_1, \dots, L_{k-1})$ has joint density $(k-1)!$ to simplify

$$= \int_{s_1}^{s_2} \frac{s^{k-1}}{(k-1)!} P(M_k \leq s^{-1}) ds \tag{3.37}$$

This proves our definition of the joint density $f_{S_k}(s)$ by definition. Noting that $P(M_k \leq s^{-1}) \in [0, 1]$ and the equality holds for all $s_1 < s_2$ also tells us that the expression is upper bounded by $\frac{s^{k-1}}{(k-1)!}$ as desired. □

For a further bound, recall from our proof above that the Jacobian of $\alpha^{-1}$ is $s^{k-1}$. Letting $\mathbb{I}_{\mathbf{A}}$ be the indicator random variable for event $\mathbf{A}$ occurring, we can write the joint density function for $(S_k, U_i/S_k)$ as

$$f(s, \ell_1, \dots, \ell_{k-1}) = s^{k-1} \mathbb{I}_{M_k < s^{-1}} \mathbb{I}_{\sum_i \ell_i \leq 1} \tag{3.38}$$

The joint density of $(L_1, \dots, L_{k-1})$ can be written

$$g(\ell_1, \dots, \ell_{k-1}) = (k-1)! \mathbb{I}_{\sum_i \ell_i \leq 1} \tag{3.39}$$

Combining the equations, we rewrite the joint density of $(S_k, U_i/S_k)$ as

$$f(s, \ell_1, \ldots, \ell_{k-1}) = \frac{s^{k-1}}{(k-1)!} g(\ell_1, \ldots, \ell_{k-1}) \mathbb{I}_{M_k < s^{-1}} \tag{3.40}$$

$$\leq \frac{s^{k-1}}{(k-1)!} g(\ell_1, \ldots, \ell_{k-1}) \tag{3.41}$$

Where the inequality comes from the fact that $\mathbb{I}_{M_k < s^{-1}}$ has a maximum of 1.

We can now replace $(S_k, U_i/S_k)$ with $(S_k, T_k)$ and $S_k$ with $U_k$, letting $f(s, t)$ be the joint density of $(S_k, T_k)$ and $g(t)$ the density of $U_k$ to get the analogous bound

$$f(s, t) \leq \frac{s^{k-1}}{(k-1)!} g(t) \tag{3.42}$$

## 3.4 Expected Number of Stable Matchings for Large n

Using Knuth's integral formulation for the probability of a matching being stable, we now provide an asymptotic analysis of the number of stable matchings one would expect as the size of the market, $n$, gets large. Specifically, we will show the following behavior, which greatly tightens the naïve $2^{n-1}$ bound we established in **Theorem 3.2**:

**Theorem 3.10 (Pittel [11]):** The expected number of stable matchings as $n \to \infty$ is asymptotic to $\frac{n \ln(n)}{e}$.

We showed earlier in this chapter that there are $n!$ possible matchings, meaning the expectation we want to bound is $n! P_n$. This section thus proves the following theorem, which immediately implies the above:

**Theorem 3.11 (Pittel [11]):** As $n \to \infty$, the probability of a matching being stable is

$$P_n = (1 + o(1))\frac{n \ln(n)}{n! e} \tag{3.43}$$

Recall that we are studying average behavior over all preference lists where each person ranks members of the opposing set independently and uniformly at random. We simulate such a random preference ordering for a symmetric size-$n$ market with $M = (m_1, \dots, m_n)$ men and $W = (w_1, \dots, w_n)$ women as follows: consider two $n \times n$ matrices of independent, identically distributed $U[0,1]$ random variables $\mathbf{X} = [X_{ij}]$ and $\mathbf{Y} = [Y_{ij}]$. A given row in $\mathbf{X}$ then represents the preference list of man $m_i$ as follows: if $X_{ij_1} < X_{ij_2} < \cdots < X_{ij_n}$ then $w_{j_n} \succ_{m_i} \cdots \succ_{m_i} w_{j_1}$. This is analogous for a given row $i$ of $\mathbf{Y}$ and the preference of woman $w_i$. Note that $\forall i, j \in [1, n]$ the random variables $X_{ij}, Y_{ij}$ are continuous and thus the probability of a tie in any given row or column is zero. Each of the $2n$ permutations for rows/columns in $\mathbf{X}, \mathbf{Y}$ are independent of each other, meaning the probability of generating a given instance of the stable marriage problem is $\frac{1}{(n!)^{2n}}$. Because the probability of any specific matching is equal to any others by this construction, we will focus on finding the probability that the matching $\mu = \{(m_i, w_i)\}_{i \in [1,n]}$ is stable.

By **Theorem 3.4**, we know that the probability of $\mu$ being stable is

$$P_n := P(\mu \text{ is stable}) = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \prod_{1 \le i \ne j \le n} (1 - x_i y_j) dx_1 \dots dx_n dy_1 \dots dy_n \tag{3.44}$$

Where $x_i = X_{ii}, y_j = Y_{jj}$. We start with an upper bound for $P_n$ by utilizing the fact that for all $\alpha \geq 0$ we have $1 - \alpha \leq e^{-\alpha - \alpha^2/2}$. Applying this to our integrand of $P_n$, we get

$$P_n = \underbrace{\int_0^1 \cdots \int_0^1}_{\text{n}} \prod_{j=1}^{n} \left( \int_0^1 e^{-ys_j - \frac{y^2 t_j}{2}} dy \right) dx_1 \ldots dx_n \tag{3.45}$$

Where we define $s_j := \sum_{i \neq j} x_i$ and $t_j := \sum_{i \neq j} x_i^2$. From our theorem statement, note that any factors whose contribution to $P_n$ is $o(\ln(n)/(n-1)!)$ are negligible and can thus be discarded. We will therefore break up our integral in 3.45 into two parts: one where $s := \sum_{i=1}^{n} x_i \leq \ln(n)$ and another where $s > \ln(n)$.

We bound the integral when $s \leq \ln(n)$ by first removing terms of the form $e^{-y^2 t_j/2}$, which are all at most 1.

$$\underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s \leq \ln(n)} \prod_{j=1}^{n} \left( \int_0^1 e^{-ys_j - \frac{y^2 t_j}{2}} dy \right) dx_1 \ldots dx_n \leq \underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s \leq \ln(n)} \prod_{j=1}^{n} \left( \int_0^1 e^{-ys_j} dy \right) dx_1 \ldots dx_n \tag{3.46}$$

Integrating, we have

$$\leq \underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s \leq \ln(n)} \prod_{j=1}^{n} \left( \frac{1 - e^{-s_j}}{s_j} \right) dx_1 \ldots dx_n \tag{3.47}$$

To further bound this expression, note that for $z > 0$,

$$\frac{d}{dz} \ln \left( \frac{1 - e^{-z}}{z} \right) = \frac{-1}{z} \left( 1 - \frac{z}{e^z - 1} \right) \tag{3.48}$$

This tells us that

$$\lim_{z\to\infty} \frac{d}{dz} \ln\left(\frac{1-e^{-z}}{z}\right) = \frac{-1}{z}(1+o(1)) \tag{3.49}$$

And also that there exists a constant $k > 0$ such that

$$\frac{d}{dz} \ln\left(\frac{1-e^{-z}}{z}\right) \in [-k, 0] \tag{3.50}$$

We now use 3.50 with our integrand from 3.47 to see that

$$\ln\left(\frac{1-e^{-s_j}}{s_j}\right) = \ln\left(\frac{1-e^{-s}}{s}\right) - \int_{s-x_j}^{x_j} \frac{d}{dz} \ln\left(\frac{1-e^{-z}}{z}\right) dz \tag{3.51}$$

$$\leq \ln\left(\frac{1-e^{-s}}{s}\right) + kx_j \tag{3.52}$$

Taking the sum over all $j \in [1, n]$ we have the bound

$$\sum_{j=1}^{n} \ln\left(\frac{1-e^{-s_j}}{s_j}\right) \leq n \ln\left(\frac{1-e^{-s}}{s}\right) + k \sum_{j=1}^{n} x_j \tag{3.53}$$

Converting the sum of natural logarithms into their product and exponentiating both sides of the inequality, this expression becomes

$$\sum_{j=1}^{n} \ln\left(\frac{1-e^{-s_j}}{s_j}\right) = \ln\left(\prod_{j=1}^{n} \frac{1-e^{-s_j}}{s_j}\right) \longrightarrow \prod_{j=1}^{n} \frac{1-e^{-s_j}}{s_j} \leq \left(\frac{1-e^{-s}}{s}\right)^{n} \cdot e^{k\sum_{j=1}^{n} x_j} \tag{3.54}$$

Here, note that in this first integral we are bounding when $s = \sum_{j=1}^{n} x_j \leq \ln(n)$, meaning we can plug this into 3.47 to further bound our original expression as

$$\leq n^k \underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s \leq \ln(n)} \left(\frac{1-e^{-s}}{s}\right)^n dx_1 \dots dx_n \tag{3.55}$$

We can now pull in **Theorem 3.9**, since the integral above is the expectation of $\left(\frac{1-e^{-S_n}}{S_n}\right)^n$ over the event $S_n \leq \ln(n)$, to get that

$$\leq \int_0^{\ln(n)} \left(\frac{1-e^{-s}}{s}\right)^n \frac{s^{n-1}}{(n-1)!} ds \tag{3.56}$$

$$= \frac{1}{(n-1)!} \int_0^{\ln(n)} \frac{(1-e^{-s})^n}{s} ds \tag{3.57}$$

$$= o\left(\frac{\ln(n)}{(n-1)!}\right) \tag{3.58}$$

With this, we've established a sufficient bound on the integral from 3.45 when $s \leq \ln(n)$, meaning we can turn our attention to the analysis for $s > \ln(n)$:

$$\underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s > \ln(n)} \prod_{j=1}^n \left(\int_0^1 e^{-ys_j - \frac{y^2 t_j}{2}} dy\right) dx_1 \dots dx_n \tag{3.59}$$

To start, we first bound a general form of the integral inside product above, using the notation $t := \sum_{i=1}^n x_i^2$ and $H(u) := \int_0^\infty z e^{-z-z^2 u/2} dz$ as follows:

$$\int_0^1 e^{-ys_j - \frac{y^2 t_j}{2}} dy \leq \frac{1}{s_j} \int_0^\infty e^{-z - \frac{z^2 t_j}{2s_j^2}} dz \tag{3.60}$$

$$\leq \frac{1}{s_j} \int_0^\infty e^{-z-\frac{z^2 t_j}{2s^2}} dz \tag{3.61}$$

$$= \frac{1}{s_j} \left( 1 - \frac{t_j}{s^2} \int_0^\infty z e^{-z-\frac{z^2 t_j}{2s_j^2}} dz \right) \tag{3.62}$$

$$\leq \frac{1}{s_j} \left( 1 - \frac{t_j}{s^2} H(t/s^2) \right) \tag{3.63}$$

$$\leq \frac{1}{s_j} e^{-\frac{t_j}{s^2} H(t/s^2)} \tag{3.64}$$

Further separating the terms in the product, we first use the fact that $\sum_{j=1}^{n} t_j = (n-1)t$ to get

$$\prod_{j=1}^{n} e^{-\frac{t_j}{s^2} H(t/s^2)} = \prod_{j=1}^{n} e^{-\frac{(n-1)t}{s^2} H(t/s^2)} \tag{3.65}$$

Moreover, we have that

$$\prod_{j=1}^{n} \frac{1}{s_j} = \frac{1}{s^n} \prod_{j=1}^{n} \left( 1 - \frac{x_j}{s} \right) \tag{3.66}$$

$$= \frac{1}{s^n} \prod_{j=1}^{n} e^{\frac{x_j}{s} + O(x_j^2/s^2)} \tag{3.67}$$

$$= \frac{1}{s^n} e^{1+O(1/\ln(n))} \tag{3.68}$$

Where the last equality is because we are working in the domain where $s > \ln(n)$. Combining these to bound the integrand in 3.59 we have,

$$\leq e^{1+O(1/\ln(n))} \underbrace{\int_0^1 \cdots \int_0^1}_{\text{n, for } s > \ln(n)} \frac{1}{s^n} e^{(n-1)\frac{t}{s^2} H(t/s^2)} dx_1 \ldots dx_n \tag{3.69}$$

Once again, we've turned our bound into one involving the expectations of functions on $U[0,1]$ random variables, meaning we can pull in **Theorem 3.9** as expressed in 3.42:

$$\leq \frac{e^{1+O(1/\ln(n))}}{(n-1)!}\left(\int_{\ln(n)}^{n} \frac{1}{s} ds\right) \mathbb{E}\left[e^{(n-1)V_n H(V_n)}\right] \tag{3.70}$$

With $V_n$ as in 3.16. Finally, we use **Lemma 3.8** along with the dominated convergence theorem on our expectation term, which gives the limit in probability

$$\lim_{n\to\infty} \mathbb{E}\left[e^{(n-1)V_n H(V_n)}\right] = \frac{1}{e^2} \tag{3.71}$$

Our integral for $s > \ln(n)$ is thus bounded

$$\leq (1+o(1))\frac{n\ln(n)}{n!e} \tag{3.72}$$

Combining with our bound on $s \leq \ln(n)$ from 3.56 we thus get our desired upper bound

$$P_n \leq (1+o(1))\frac{n\ln(n)}{n!e} \tag{3.73}$$

What's left is to now bound $P_n$ from below, which will give us our final equality. Choose $\epsilon \in (0,1)$ and denote by $D(\epsilon) \subseteq [0,1]^n$ the subset with $\mathbf{x} = (x_1, \ldots, x_n)$ such that

$$3\ln(n) \leq s \leq \frac{n}{\ln^2(n)} \tag{3.74}$$

$$\frac{x_j}{s} \leq (1+\epsilon)\frac{\ln(n)}{n} \tag{3.75}$$

$$\frac{t}{s^2} \leq (1+\epsilon)\frac{2}{n} \tag{3.76}$$

Combining the first two of these inequalities, we have that

$$x_j \leq \frac{1+\epsilon}{\ln(n)} < 1 \tag{3.77}$$

We now note that defining $P_n(\epsilon)$ as the part of $P_n$ contributed by the region $D(\epsilon)$, we have by definition $P_n \geq P_n(\epsilon)$. As an integral analogous to 3.44, this is

$$P_n(\epsilon) := \int_{x \in D(\epsilon)} \prod_{j=1}^{n} \left( \int_0^1 \prod_{i \neq j} (1 - x_i y_j) dy_j \right) dx_1 \ldots x_n \leq P_n \tag{3.78}$$

We start with the innermost product, which we can bound by combining 3.77 with the fact that $\lim_{x \to 0} 1 - x = e^{-x - \frac{x^2}{2} + O(x^2)}$ to get, for $\gamma_n = \frac{c}{\ln(n)}$ with $c > 0$, that

$$\prod_{i \neq j} (1 - x_i y_j) \geq e^{-y_j s_j - y_j^2 t_j \frac{1+\gamma_n}{2}} \tag{3.79}$$

For each $j \in [1, n]$ we can now bound the whole inner integral as

$$\int_0^1 \prod_{i \neq j} (1 - x_i y_j) dy_j \geq \int_0^1 e^{-y s_j - y^2 t_j \frac{1+\gamma_n}{2}} dy \tag{3.80}$$

Letting $z = y s_j$ and using Jensen's inequality, we have

$$= \frac{1}{s_j} \int_0^1 e^{-z} e^{-z^2 \frac{t_j}{s_j^2} \frac{1+\gamma_n}{2}} dz \tag{3.81}$$

$$\geq \frac{1 - e^{-s_j}}{s_j} e^{-\frac{t_j}{s_j^2} \frac{1+\gamma_n}{2(1-e^{-s_j})} \int_0^{s_j} z^2 e^{-} z dz} \tag{3.82}$$

Using 3.77, which was a direct result of our definition of $D(\epsilon)$, we have

$$s_j = s - x_j \tag{3.83}$$

$$= se^{-\frac{x_j}{s}+O(x_j^2/s^2)} \tag{3.84}$$

$$= se^{-\frac{x_j}{s}+o(x_j/s)} \tag{3.85}$$

$$\geq c\ln(n) \quad \forall c \in (2,3) \tag{3.86}$$

Uniformly over all $\mathbf{x} \in D(\epsilon)$ for large $n$. This tells us that

$$1 - e^{s_j} = 1 + O(n^{-c}) \tag{3.87}$$

$$\int_0^{s_j} z^2 e^{-z} dz = 2 - 2e^{-s_j}\left(1 + s_j + \frac{s_j^2}{2}\right) \tag{3.88}$$

$$= 2 + O(n^{-c'}) \text{ for } c' = c - 2 \tag{3.89}$$

Combining this with our bound from 3.82, we get

$$\geq \frac{1 + O(n^{-c'})}{s} e^{\frac{x_j}{s}+o(x_j/s)} e^{-\frac{t_j}{s_j^2}(1+O(n/\ln(n)))} \tag{3.90}$$

Using the fact that $s_j \leq s$ and $\sum_{j=1}^{n} t_j = (n-1)t$, we bound the full outer integrand in 3.78 uniformly over all $\mathbf{x} \in D(\epsilon)$ as

$$\prod_{j=1}^{n}\left(\int_0^1 \prod_{i \neq j}(1 - x_i y_j) dy_j\right) \geq es^{-n} e^{-n\frac{t_j}{s_j^2}(1+O(n/\ln(n)))} \tag{3.91}$$

We now consider the following change of variables

$$\alpha\colon (x_1, \ldots, x_n) \to (s, \ell_1, \ldots, \ell_{n-1}) \tag{3.92}$$

Where $\ell_i = \frac{x_i}{s}$, $\ell_n = \frac{x_n}{s}$, and by construction $\sum_{i=1}^{n} \ell_i = 1$ We've actually seen this transformation before in 3.34, where we noted that the Jacobian of the inverse transformation is $s^{n-1}$. With this transformation, our equations for defining $D(\epsilon)$ from 3.74 become

$$3\ln(n) \leq s \leq \frac{n}{\ln^2(n)} \tag{3.93}$$

$$\ell_j \leq (1+\epsilon)\frac{\ln(n)}{n} \tag{3.94}$$

$$\sum_{j=1}^{n} \ell_j^2 \leq (1+\epsilon)\frac{2}{n} \tag{3.95}$$

Letting $D(\epsilon)_\ell$ be the region in $[0,1]^n$ defined by 3.94 and 3.95, we see that $D(\epsilon)$ can now be thought of as the direct product of the region defined by 3.93 with $D(\epsilon)_\ell$. Using this fact along with our bound from 3.91, we can write our full integral for $P_n(\epsilon)$ from 3.78 as

$$P_n(\epsilon) \geq \frac{e}{(n-1)!}\left(\int_{3\ln(n)}^{n/\ln^2(n)} \frac{1}{s} ds\right)\left(\int_{D(\epsilon)_\ell} (n-1)! e^{-n(1+O(n/\ln(n)))\sum_{j=1}^{n}\ell_j^2} d\ell_1 \ldots d\ell_{n-1}\right) \tag{3.96}$$

Looking at the first integral term, we can see that it is asymptotically equal to $\ln(n)$. Focusing on the second term, 3.95 tells us that the integrand over $D(\epsilon)_\ell$ is at least

$$(n-1)! e^{-2(1+\epsilon)(1+O(1/\ln(n)))} \tag{3.97}$$

Finally, using **Theorem 3.5**, where we showed that $(n-1)!$ is the joint density of the first

$(n-1)$ subintervals $L_i$ over $[0,1]$, we can write

$$P_n(\epsilon) \geq \frac{e}{(n-1)!} \cdot \ln(n) \cdot e^{-2} \cdot P(\mathcal{L}_n) = \frac{n\ln(n)}{n!e} \tag{3.98}$$

Where $\mathcal{L}_n$ is the event that $\max_{j\in[n]} L_i \leq (1+\epsilon)\frac{\ln(n)}{n}$ where $\sum_{j=1}^{n} L_j^2 \leq (1+\epsilon)\frac{2}{n}$. We can use **Lemma 3.6** here, which tells us that

$$\lim_{n\to\infty} P\left(\max_{j\in[n]} L_i \geq \frac{(1+\epsilon)\ln(n)}{n}\right) = 0 \tag{3.99}$$

Combined with **Lemma 3.8**, we have that

$$\lim_{n\to\infty} P(\mathcal{L}_n) = 1 \tag{3.100}$$

Therefore, as $n \to \infty$ and $\epsilon \to 0$ we get our desired lower bound

$$P_n \geq P_n(\epsilon) \geq \frac{n\ln(n)}{n!e} \tag{3.101}$$

Which concludes our proof of **Theorem 3.11**. □

# 4

# Probable Ranks in a Stable Matching

Our work in Chapter 3 revolved around characterizing the space of possible stable matchings, with a focus on finding the expected number of such outcomes in a size $n$ marriage market. This chapter builds upon those results in an orthogonal direction—describing a measure of how *desirable* a particular matching is for both sides of the market. By studying the asymptotic behavior of the minimum and maximum total ranks from Pittel [11] and [12], we show that the side of the market which proposes in our deferred acceptance algorithm [4] obtains a significantly more favorable outcome as a group.

## 4.1 The Rank of a Matching

Recall our general marriage market formulation, which has size $n$ sets of men $M$ and women $W$, each element of which has a complete preference list over the other set. Consider the preferences of man $m_i \in M$, which are some permutation $\pi$ over the set of women

$$P(m_i) = w_{\pi(1)}, \ldots, w_{\pi(n)} \tag{4.1}$$

Then, if a given matching $\mu$ assigns $\mu(m_i) = w_{\pi(k)}$, his $k^{th}$ choice, then we say that the rank of the matching for $m_i$ is $k$. The rank of the overall matching for men, $Q(\mu)$, can then be computed as the sum of the ranks for each individual. The same definition applies to the woman side of the market as well. Thus, we see that there exists a trivial lower and upper bound on the rank of a matching: $n, n^2$ respectively. The former occurs when every man is matched to his top choice woman, giving an overall rank of $n \times 1$ while the latter occurs when every man is matched to his last choice woman, giving rank $n \times n$. In the rest of this chapter, we will tighten both of these bounds and better characterize the expected minimum and maximum total ranks for a stable marriage.

## 4.2 Probability of a Stable Marriage of Given Rank

To start, however, we once again return to **Theorem 3.4**, which was crucial in our Chapter 3 bounds. Instead of simply determining the probability of a random matching being stable, however, we will go further and find an analogous expression for the probability of a random matching being stable and having a specific rank. Letting $P_{nk}$ for $k \in [n, n^2]$ be the probability that a random matching in a size $n$ market has rank $k$ and is stable, we prove:

**Theorem 4.1 (Pittel [12]):** For $x, y, z \in [0,1]$ and $i, j \in [1, n] \in \mathbb{Z}$

$$P_{nk} = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \left[z^{k-n}\right] \prod_{1 \leq i \neq j \leq n} (1 - x_i(1 - z + zy_j)) dx_1 \dots dx_n dy_1 \dots dy_n \tag{4.2}$$

Where $\left[z^{k-n}\right]$ denotes the coefficient of $z^{k-n}$ in the product inside the integrand.

*Proof:* As we did for the proof of **Theorem 3.11**, we generate a random ranking system for $M = (m_1, \dots, m_n)$ men and $W = (w_1, \dots, w_n)$ women as follows: consider two $n \times n$ matrices of independent, identically distributed $U[0,1]$ random variables $\mathbf{X} = [X_{ij}]$ and $\mathbf{Y} = [Y_{ij}]$. A given row in $\mathbf{X}$ then represents the preference list of man $m_i$ as follows: if $X_{ij_1} < X_{ij_2} < \cdots < X_{ij_n}$ then $w_{j_n} \succ_{m_i} \cdots \succ_{m_i} w_{j_1}$. This is analogous for a given row $i$ of $\mathbf{Y}$ and the preference of woman $w_i$. Because the probability of any specific matching is equal to any other by this construction, without loss of generality we will consider the matching $\mu = \{(m_i, w_i)\}_{i \in [1,n]}$.

The rank $Q(\mu)$ of this matching is given by

$$Q(\mu) = n + \sum_{i=1}^{n} |\{j|X_{ij} < X_{ii}\}| \tag{4.3}$$

$P_{nk}$ is then the probability of the event $A = \{\mu \text{ is stable and } Q(\mu) = k\}$. Additionally, define $P_{nk}(x, y)$ to be the conditional probability $P(A|X_{ii} = x_i, Y_{jj} = y_j)$ for $i, j \in [1, n]$. Note that by construction, all entries $X_{ij}$, $Y_{ij}$ are independent from each other, meaning to prove 4.2, we only need to show that $P_{nk}(x, y)$ is equal to the integrand, after which Fubini's theorem gives the desired conclusion over all $i, j \in [1, n]$.

Letting $\mathbb{I}_\mu$ be the indicator random variable for the event $\mu$ is stable, we can write our probability as the expectation

$$P_{nk}(x, y) = \left[z^k\right] \mathbb{E}\left[\mathbb{I}_\mu \cdot z^{Q(\mu)}|X_{ii} = x_i, Y_{jj} = y_j\right] \tag{4.4}$$

We can reformulate the expectation in a way that makes evaluation much more intuitive: $z^{Q(\mu)}$ can be thought of as a probability $z$ event occuring $Q(\mu)$ times. By the definition of rank in 4.3, we have that $\sum_{i=1}^{n} |\{j|X_{ij} < X_{ii}\}| = Q(\mu) - n$. Fix $z \in (0, 1)$ and note that each $X_{ij}$ is independent. Consider the following: independently for every entry $X_{ij} \in \mathbf{X}$, if $X_{ij} < X_{ii} = x_i$ then label $(i, j)$ as 1 with probability $z$ and 0 with probability $1 - z$. Let $D$ be the event that for a given matrix, all entries $X_{ij} < X_{ii}$ are marked as 1, meaning $P(D) = z^{Q(\mu)-n}$. Therefore,

$$\mathbb{E}\left[\mathbb{I}_\mu \cdot z^{Q(\mu)}|X_{ii} = x_i, Y_{jj} = y_j\right] = z^n \cdot P(\mu \text{ is stable and } D|X_{ii} = x_i, Y_{jj} = y_j) \tag{4.5}$$

For notation, let $\mathcal{B} = \{\mu \text{ is stable and } D\}$, meaning the above expectation is

$$= z^n \cdot P(\mathcal{B}|X_{ii} = x_i, Y_{jj} = y_j) \tag{4.6}$$

For a given $(i, j), i \neq j$ let $\mathcal{B}_{ij}$ then be the event

$$X_{ii} < X_{ij} \text{ or } X_{ii} > X_{ij}, Y_{jj} < Y_{ij}, (i, j) \text{ is labelled } 1 \tag{4.7}$$

With this definition, we have that $\mathcal{B} = \bigcap_{i \neq j} \mathcal{B}_{ij}$. The events $\mathcal{B}_{ij}$, however, are independent when conditioned on $X_{ii} = x_i, Y_{jj} = y_j$, meaning their conditional probability is

$$P(\mathcal{B}_{ij}|X_{ii} = x_i, Y_{jj} = y_j) = (1 - x_i) + x_i(1 - y_j)z \tag{4.8}$$

The probability of $\mathcal{B}$ is then

$$P(\mathcal{B}|X_{ii} = x_i, Y_{jj} = y_j) = \prod_{i \neq j \in [1,n]} (1 - x_i(1 - z + zy_j)) \tag{4.9}$$

Combining this with our results from equations 4.4, 4.5, and 4.6 we get our desired result

$$P_{nk}(x, y) = \left[z^{k-n}\right] \prod_{i \neq j \in [1,n]} (1 - x_i(1 - z + zy_j)) \tag{4.10}$$

Thus, $P_{nk}(x, y)$ equals the integrand in our theorem statement 4.2, meaning by Fubini's theorem we have our desired equality. □

## 4.3 Asymptotic Bound for the Total Rank

Define by $r_n$ the minimum total rank for any stable matching and $R_n$ the maximum such rank, both considered from the male side of the market. In this section, we combine our insights from the previous parts of this chapter in order to find the convergence of $r_n, R_n$ in probability as $n \to \infty$. To start, we prove the following theorem:

**Theorem 4.2:** For all $\alpha > 0$ and $\delta \in (0, e^{\alpha} - 1)$,

$$P(r_n \geq n(\ln(n) - \ln(\ln(n)) - \alpha)) \geq 1 - O(n^{-\delta}) \tag{4.11}$$

$$P(R_n \leq n^2 \ln^{-1}(n)(1 + \ln^{-1}(n)(\ln(\ln(n)) + \alpha))) \geq 1 - O(n^{-\delta}) \tag{4.12}$$

*Proof:* We start by recalling our expression from **Theorem 4.1** for the probability $P_{nm}$ that a stable matching of size $n$ has rank $m$:

$$P_{nm} = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \left[z^{m-n}\right] \prod_{1 \leq i \neq j \leq n} (1 - x_i(1 - z + zy_j))dx_1 \ldots dx_n dy_1 \ldots dy_n \tag{4.13}$$

Additionally, let $\Phi(x, y, z) = \prod_{1 \leq i \neq j \leq n}(1 - x_i(1 - z + zy_j))$, a convention we will use for this proof to simplify the notation:

$$P_{nm} = \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \left[z^{m-n}\right] \Phi(x, y, z) d\mathbf{x} d\mathbf{y} \tag{4.14}$$

For each $k \in [n, n^2]$, which are the bounds for total rank, we can write

$$P(r_n \leq k) \leq n! \sum_{m=n}^{k} P_{nm} \qquad\qquad P(R_n \geq k) \leq n! \sum_{m=k}^{n^2} P_{nm} \tag{4.15}$$

We can now apply Chernoff bounds [3], where inf refers to the infimum of a set, to get

$$P(r_n \leq k) \leq n! \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \inf_{z \in (0,1]} \left( z^{n-k} \Phi(x, y, z) \right) d\mathbf{x} d\mathbf{y} \tag{4.16}$$

$$P(R_n \geq k) \leq n! \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \inf_{z \geq 1} \left( z^{n-k} \Phi(x, y, z) \right) d\mathbf{x} d\mathbf{y} \tag{4.17}$$

**Part I:** The following part of the proof will focus on the $r_n$ bound. We start by noting that $(1 - x_i(1 - z + zy_j))$ can be bounded by $e^{(-x_i(1-z+zy_j))}$. Applying this and integrating our expression in 4.16 with respect to $\mathbf{y}$, we get

$$P(r_n \leq k) \leq n! \underbrace{\int_0^1 \cdots \int_0^1}_{2n} \inf_{z \in (0,1]} \left( z^{n-k} e^{s(z-1)(n-1)} \prod_{j=1}^{n} \frac{1 - e^{-zs_j}}{zs_j} \right) d\mathbf{x} \tag{4.18}$$

Where the simplification is analogous to the one performed in our proof of **Theorem 3.11** and $s = \sum_i^n x_i$ and $s_j = \sum_{i \neq j} x_i$ are defined identically. Using the results we obtained in 3.48 to 3.50 and the fact that $z \leq 1$, we know for a constant $c$ that

$$\prod_{j=1}^{n} \frac{1 - e^{-zs_j}}{zs_j} \leq c \left( \frac{1 - e^{-zs}}{zs} \right)^n \tag{4.19}$$

Moreover, **Theorem 3.9** then tells us that for $H(s, z) := s(z - 1)(n - 1) + n \ln(1 - e^{-zs}) -$

$k\ln(z) - \ln(s)$ we have

$$P(r_n \leq k) \leq cn \int_0^n \inf_{z\in(0,1]} e^{H(s,z)} ds \;\; \forall k \in [n, n^2] \tag{4.20}$$

Now note that looking at our theorem statement in 4.11 we want to show that for $k = n(\ln(n) - \ln(\ln(n)) - \alpha)$, the right hand side of 4.20 goes to 0 as $n^{-\delta}$ for $\delta \in (0, e^{\alpha} - 1)$. To do this, we first look at our expression $H(s,z)$ for $zs = b$ where $b$ satisfies

$$h(\beta) = k \text{ for } h(\beta) := \beta \cdot \left((n-1) + \frac{n}{e^{\beta} - 1}\right) \tag{4.21}$$

With this condition, we have

$$H_z := s(n-1) + \frac{ns}{e^{zs} - 1} - \frac{k}{z} = 0 \tag{4.22}$$

In addition, we have that

$$h'(\beta) \geq \lim_{\beta \to 0^+} h'(\beta) = \frac{n}{2} - 1 \tag{4.23}$$

Meaning that 4.21 has a unique positive root $b$, which, for $k$ defined as above, is

$$b = \frac{k}{n}\left(1 + O(\ln(n)/n)\right) \tag{4.24}$$

$$= \ln(n) - \ln(\ln(n)) - \alpha + O\left(\frac{\ln^2(n)}{n}\right) \tag{4.25}$$

$$< \ln(n) - \ln(\ln(n)) - \alpha' \text{ for all } \alpha' < \alpha \tag{4.26}$$

This allows us to simplify the integral in 4.20, noting that if $s \leq b$ we can let $z = 1$ and otherwise let $z = \frac{b}{s}$, meaning

$$\int_0^n \inf_{z \in (0,1]} e^{H(s,z)} ds \leq \int_0^b \frac{(1-e^{-s})^n}{s} ds + \frac{(1-e^{-b})^n}{b^k} e^{b(n-1)} \int_b^\infty \frac{s^{k-1}}{e^{s(n-1)}} ds \tag{4.27}$$

$$\leq \int_0^b \frac{(1-e^{-s})^n}{s} ds + \frac{(1-e^{-b})^n}{(b(n-1))^k} e^{b(n-1)} (k-1)! \tag{4.28}$$

For ease of notation, we will refer to the first term as $T_1$ and the second as $T_2$. Then,

$$T_1 \leq b(1-e^{-b})^{n-1} \tag{4.29}$$

$$= O\left(be^{-ne^{-b}}\right) \tag{4.30}$$

$$= O\left(n^{e^{-\alpha'}}\right) \text{ for } \alpha' \in [0, \alpha] \tag{4.31}$$

Which is our desired bound. We can now focus our efforts on $T_2$, where we utilize Stirling's formula with the functions

$$\Gamma_n(k) := F_n(k, b) \tag{4.32}$$

$$F_n(\kappa, \beta) := \beta(n-1) + n \ln(1-e^{-\beta}) - \kappa \ln(\beta) + (\kappa - 1) \ln\left(\frac{\kappa - 1}{e(n-1)}\right) \tag{4.33}$$

to get

$$T_2 = O\left(e^{\Gamma_n(k)} \sqrt{\frac{\ln(n)}{n}}\right) \tag{4.34}$$

We want to find an upper bound on $e^{\Gamma_n(k)}$, meaning we first turn our focus to $F_n(\kappa, \beta)$.

$$\left.\frac{dF_n(\kappa,\beta)}{d\beta}\right|_{\beta=b} = (n-1) + \frac{n}{e^{\beta}-1} - \left.\frac{\kappa}{\beta}\right|_{\beta=b} = 0 \tag{4.35}$$

$$= \frac{dF_n(\kappa,\beta)}{d\beta} = \ln\left(\frac{\kappa-1}{\beta(n-1)}\right) \tag{4.36}$$

This means that we have a stationary point $(\kappa_0, \beta_0)$ of $F_n(\kappa, \beta)$ defined by

$$\kappa_0 = \beta_0(n-1)+1 \tag{4.37}$$

$$e^{\beta_0} - 1 = n\beta_0 \tag{4.38}$$

Solving for this point, we get that

$$\kappa_0 = n(\ln(n) + \ln(\ln(n))(1 + O(1/\ln(n)))) \tag{4.39}$$

$$\beta_0 = \ln(n) + \ln(\ln(n))(1 + O(1/\ln(n))) \tag{4.40}$$

Plugging these values in to our original equation 4.32 we get

$$\Gamma_n(\kappa_0) = F_n(\kappa_0, \beta_0) = n\ln(1-e^{-\beta_0}) - \ln(\beta_0) \tag{4.41}$$

$$= -\ln(\beta_0) + O(ne^{-\beta_0}) \tag{4.42}$$

$$= -\ln(\beta_0) + O(1/\ln(n)) \tag{4.43}$$

We will now show that $\Gamma$ evaluated at $\kappa_0$ is actually equal to the maximum value of $\Gamma$ over all $\kappa$. To do this, we use 4.35 and 4.36 to get that

$$\frac{d}{d\kappa}\Gamma_n(\kappa) = \left.\frac{dF_n(\kappa,\beta)}{d\beta}\right|_{\beta=b} \tag{4.44}$$

$$= \ln\left(1 + \frac{1}{\beta(n-1)}\left(\frac{\beta n}{e^{\beta}-1} - 1\right)\right) \tag{4.45}$$

Where we note that because $k < \kappa_0$ this final expression is less than 0, meaning $\Gamma_n(k) < \Gamma_n(\kappa_0)$. In addition, looking closely at our definition of $k$ and expression for $\kappa_0$, we see that their difference, $\kappa_0 - k$ should be close to $2n\ln(\ln(n))$. Choosing $\kappa_1, \beta_1$ such that

$$\frac{n\beta_1}{e^{\beta_1}-1} = \ln(\ln(n)) \tag{4.46}$$

We get, analogous to our result in 4.40, that

$$\beta_1 = \ln(n) + \ln(\ln(n))(1 + O(1/\ln(n))) \tag{4.47}$$

Then, comparing our equations for $k, \kappa_0, \kappa_1$ we have that $\kappa_1 \in (k, \kappa_0)$, allowing us to bound

$$\Gamma_n(k) = \Gamma_n(\kappa_0) - \int_k^{\kappa_0} \Gamma_n'(\kappa)d\kappa \tag{4.48}$$

$$\leq \Gamma_n(\kappa_0) - \int_k^{\kappa_1} \Gamma_n'(\kappa)d\kappa \tag{4.49}$$

Using our expression from 4.45 construction in 4.46 to bound $\frac{\beta n}{e^{\beta}-1}$, for all $\kappa \in [k, \kappa_1]$:

$$\frac{d}{d\kappa}\Gamma_n(\kappa) = (1 + o(1))\frac{1}{e^{\beta}} \tag{4.50}$$

Taking the derivative of $\kappa$ with respect to $b$, we have

$$\frac{d\kappa}{db} = (n-1) + \frac{n(e^b - be^b - 1)}{(e^b-1)^2} \tag{4.51}$$

Allowing us to now combine our previous equations to see that

$$\Gamma_n(k) \leq \Gamma_n(\kappa_0) - (1-o(1))n(e^{-b} - e^{-\beta_1}) \tag{4.52}$$

$$\leq -e^{\alpha'} \ln(n) \text{ for all } \alpha' \in (0, \alpha) \tag{4.53}$$

Where we get 4.53 by bounding $ne^{-b} \geq e^{\alpha'}$ as in 4.29 and bounding $ne^{-\beta_1} = O(1/\ln(n))$ using 4.47. Thus, we have that $T_2$ is bounded as

$$T_2 = O\left(n^{-1/2-e^{\alpha'}}\right) \tag{4.54}$$

Combining these results and our definition of $k$, we thus have that

$$P(r_n \leq n(\ln(n) - \ln(\ln(n)) - \alpha)) = O\left(n^{-e^{\alpha'}+1}\right) \tag{4.55}$$

Which gives our desired bound.

**Part II:** We now turn our attention to the $R_n$ bound. Analogously to our $r_n$ proof, we start by noting that for a constant $c$,

$$Pr(R_n \geq k) \leq cn \int_0^n \inf_{z\geq 1} e^{H_1(s,z)} P(M_n \leq s^{-1}) ds \tag{4.56}$$

Where we define for constants $f$ and $s_0$,

$$H_1(s,z) = \begin{cases} H(s,z) + fsz & s \leq s_0 \\ H(s,z) & s > s_0 \end{cases} \tag{4.57}$$

As we did for $r_n$, looking at our theorem statement in 4.12 we want to show that for $k = \frac{n^2}{\ln(n)}(1-(\ln(\ln(n))+\alpha)\ln^{-1}(n))$, the right hand side of 4.56 goes to $0$ as $n^{-\delta}$ for $\delta \in (0, e^{\alpha}-1)$. The unique root of 4.21 is now

$$b = \frac{n}{\ln(n)}\left(1 + \frac{\ln(\ln(n)) + \alpha_n}{\ln(n)}\right) \text{ where } \alpha_n = \alpha + o(1) \tag{4.58}$$

In addition, defining

$$b_1 = \frac{n}{\ln(n)}\left(1 + \frac{\ln(\ln(n)) + \alpha'}{\ln(n)}\right) \text{ where } \alpha' \in (0, \alpha) \tag{4.59}$$

This allows us to simplify the integral in 4.56, noting that if $s \geq b$ we can let $z = 1$ and otherwise let $z = \frac{b}{s}$, meaning for

$$T_1 = e^{fb}\int_0^{b_1} s^{k-1}e^{-s(n-1)}P(M_n \leq s^{-1})ds \tag{4.60}$$

$$T_2 = \int_{b_1}^{b} s^{k-1}e^{-s(n-1)}P(M_n \leq s^{-1})ds \tag{4.61}$$

$$T_3 = \int_{b}^{n} \frac{1}{s}(1-e^{-s})^n P(M_n \leq s^{-1})ds \tag{4.62}$$

We can bound our desired integral as

$$\int_0^n \inf_{z\geq 1} e^{H_1(s,z)} P(M_n \leq s^{-1}) ds \leq \frac{1}{b^k}(1-e^{-b})^n e^{b(n-1)}(T_1+T_2)+T_3 \tag{4.63}$$

We start with $T_3$ since it's the simplest to analyze. When $s \in [b, n]$ we have that $P(M_n \leq s^{-1}) \leq P(M_n \leq b^{-1})$ by the definition of $b$ from 4.58. We then have, by our bounds from Chapter 3, that

$$T_3 \leq P(M_n \leq b^{-1}) \int_b^n \frac{1}{s} ds = O\left(n^{-e^{\alpha_3}}\right) \text{ for all } \alpha_3 \in (0, \alpha) \tag{4.64}$$

Now we bound $T_2$, starting with a similar bound of

$$T_2 \leq P(M_n \leq b_1^{-1}) \int_{b_1}^b s^{k-1} e^{-s(n-1)} ds \tag{4.65}$$

Looking at each term separately, once again using the inverse $b^{-1}$ of 4.58,

$$P(M_n \leq {b_1}^{-1}) = O\left(n^{-e^{\alpha_2}}\right) \text{ for all } \alpha_2 \in (0, \alpha') \tag{4.66}$$

In addition, the integral can be bounded from above by

$$\int_{b_1}^b s^{k-1} e^{-s(n-1)} ds \leq \int_0^\infty s^{k-1} e^{-s(n-1)} ds \tag{4.67}$$

$$= \frac{(k-1)!}{(n-1)^k} \tag{4.68}$$

$$= O\left(\left(\frac{k-1}{e(n-1)}\right)^{k-1}\right) \tag{4.69}$$

Where we use the fact that from our definition of $k$, $k$ is dominated by $\frac{n^2}{\ln(n)}$ for large $n$. Therefore, we have that

$$T_2 = O\left(n^{-e^{\alpha_2}}\left(\frac{k-1}{e(n-1)}\right)^{k-1}\right) \tag{4.70}$$

Finally, we get to $T_1$, which for which we start with the similar first bound

$$T_1 \leq e^{fb}\int_0^{b_1} s^{k-1}e^{-s(n-1)}ds \tag{4.71}$$

$$= e^{fb}\int_0^{b_1} e^{\ln(s)(k-1)-s(n-1)}ds \tag{4.72}$$

Define the power of the exponent as $\varphi(s) = \ln(s)(k-1) - s(n-1)$. Then, the maximum of $\varphi(s)$ is $s^* = \frac{k-1}{n-1}$. Plugging in our definition for $k$, we get

$$s^* = \frac{n}{\ln(n)}\left(1 + (\ln(\ln(n)) + \alpha^*)\ln^{-1}(n)\right) \text{ where } \alpha^* = \alpha + o(1) \tag{4.73}$$

$$= b_1(1 + o(1)) \tag{4.74}$$

Where we get the second equality by comparing to our definition of $b_1$ in 4.59. Now, because $\frac{d^2x}{dx^2}\varphi(s) = \frac{1-k}{s^2}$ we can write, for $s' \in [b_1, s^*]$, that

$$T_1 \leq b_1 e^{fb}e^{\varphi(b_1)} \tag{4.75}$$

$$\leq b_1 e^{fb}e^{\varphi(s^*)+\frac{1}{2}\varphi''(s')(b_1-s^*)^2} \tag{4.76}$$

$$= O\left(\left(\frac{k-1}{e(n-1)}\right)^{k-1} e^{\frac{-gn^2}{2\ln^3(n)}}\right) \text{ for } g > 0 \tag{4.77}$$

Combining our bounds for $T_1, T_2, T_3$ with 4.63 we then have

$$P(R_n \leq k) \leq cn(n^{-e^{\alpha_2}} e^{\Gamma_n(k)} + n^{-e^{\alpha_3}}) \tag{4.78}$$

Which gives our desired bound since we showed in 4.43 that $\Gamma_n(k)$ has a maximum that is less than $0$. $\square$

This theorem actually does the brunt of the work for our eventual desired result, which is the asymptotic convergence of $r_n, R_n$. Specifically, we will leverage **Theorem 4.2** to show the following:

**Theorem 4.3:** The limit in probability as $n \to \infty$ of $r_n, R_n$ is

$$r_n \to n \ln(n) \qquad\qquad R_n \to \frac{n^2}{\ln(n)} \tag{4.79}$$

*Proof:* We will pull in a few important results within the field. First, recall that the man-proposing deferred acceptance algorithm of Gale and Shapley [4] resulted in the man-optimal stable matching, meaning the overall rank of matches for men was minimized. McVitie and Wilson [9] provided an analogous algorithm in which each round consists of a single proposal, meaning the minimum rank $r_n$, which is the rank generated by the male-optimal algorithm, can be computed as the number of steps in their algorithm. Knuth [6] and Wilson [20] then showed that this is dominated by the number of draws in the $n-$size coupon collector problem. We thus start by formalizing this problem and determining the expected number of draws.

The coupon collector problem asks, given $n$ coupons, how many times do you expect to draw with replacement before having drawn each coupon at least once? To answer this, we construct random variables $D, d_i$ which measure the number of draws to collect all $n$ coupons and the number of draws it takes to draw the $i^{th}$ coupon after $i-1$ unique ones have been seen, respectively. The probability of collecting a new coupon after $i-1$ have been seen is

$$p_i = \frac{n-(i-1)}{n} = \frac{n-i+1}{n} \tag{4.80}$$

Then, $d_i$ follows a geometric distribution with parameter $p_i$. Using linearity of expectation, we have that

$$\begin{aligned}
\mathbb{E}[D] &= \sum_{i=1}^{n} \mathbb{E}[d_i] && (4.81)\\
&= \sum_{i=1}^{n} \frac{1}{p_i} && (4.82)\\
&= \sum_{i=1}^{n} \frac{n}{n-i+1} && (4.83)\\
&= n\sum_{i=1}^{n} \frac{1}{n-i+1} && (4.84)\\
&= nH_n && (4.85)
\end{aligned}$$

Where $H_n$ is the $n^{th}$ harmonic number, defined as $1+\frac{1}{2}+\cdots+\frac{1}{n}$. For $\epsilon > 0$ let $A$ be the event that a particular coupon hasn't been seen in the first $N=(1+\epsilon)n\ln(n)$ draws. Then,

$$P(r_n \leq (1+\epsilon)n\ln(n)) \geq 1-nP(A) = 1-n\left(1-\frac{1}{n}\right)^N \tag{4.86}$$

$$\geq 1 - ne^{\frac{-N}{n}} = 1 - n^{-\epsilon} \tag{4.87}$$

Meaning the limit of $P(r_n \leq (1+\epsilon)n\ln(n))$ approaches 1 as $\epsilon\ln(n)$ approaches $\infty$. We can combine this with our result for $r_n$ in **Theorem 4.2** to get that for $\alpha > 0$ and $\omega \to \infty$,

$$P\left(r_n \in [n(\ln(n) - \ln(\ln(n)) - \alpha), n(\ln(n) + \omega)]\right) \to 1 \tag{4.88}$$

Meaning $\frac{r_n}{n\ln(n)}$ approaches 1 in probability as desired.

We now move on to the $R_n$ bound. Once again considering McVitie and Wilson's man-proposing algorithm, let $z_i$ be the number of proposals received by woman $w_i$ and let $y_i$ be the rank of her partner according to her preference list. We have

$$\mathbb{E}[z_1] = \cdots = \mathbb{E}[z_n] \leq \frac{1}{n}\mathbb{E}[D] = H_n \tag{4.89}$$

Conditioning on $z_i = k$, the rank $y_i$ is distributed as $Bin(n-k, \frac{1}{k+1})$, meaning

$$\mathbb{E}[y_i|z_i] = 1 + \frac{n - z_i}{z_i + 1} = \frac{n+1}{z_i+1} \tag{4.90}$$

Plugging in 4.89 and using Jensen's inequality, we have

$$\mathbb{E}[y_i] \geq \frac{n+1}{\mathbb{E}[z_i]+1} = \frac{n+1}{H_n+1} \tag{4.91}$$

Now, linearity of expectation tells us that

$$\mathbb{E}[R_n] = \sum_{i=1}^{n} \mathbb{E}[y_i] \geq \frac{n(n+1)}{H_n+1} = \frac{n^2}{\ln(n)}(1 + O(1/\ln(n))) \tag{4.92}$$

Now, similarly to our analysis for $r_n$, we can bring in **Theorem 4.2** to get that $\frac{R_n \ln(n)}{n^2}$ approaches 1 in probability. □

This result should be quite in line with our intuition about the marriage market: when men propose, then generated stable match is quite favorable to them as a whole. From **Theorem 4.3**, however, we see that the extent to which this initiative pays off is drastic: since $R_n$, the maximum male-sided rank for a stable matching corresponds to the woman-proposing algorithm, the side that proposes has, in expectation, an $\frac{n}{\ln^2(n)}$ factor improvement in rank.

# 5 Conclusion

In this thesis, we set out to explore the asymptotic structure of the space of stable matchings in symmetric markets. Specifically, we showed important bounds from Pittel [11], [12] on the expected number of stable matchings and the expected minimum and maximum rank of stable matchings. This analysis provides a different way of looking at marriage markets, with a focus on average behavior over large sets instead of on specific, game-theoretic concerns.

In addition, these complex bounds were supported by an exploration in more fundamental probability theory, with a particular emphasis on how we can characterize partitions of the unit interval. Moreover, in order to fully understand the rich bounds of Pittel and Knuth [6], we built up foundational knowledge and intuition in stable matching, with a focus on key historical moments, important applications, and interesting mathematics.

Moving forward, the work in this thesis can be easily extended by considering the two implicit assumptions made throughout our proofs: that the market we are dealing with is symmetric and that preferences are complete and strictly ranked. In many real-world markets, designers have to deal with allocating goods between or matching among sets of people that are vastly different in size, where it may not even be feasible for an individual to have preferences over all items. In particular, the basic deferred acceptance protocol fails to account for instances where two potential partners may be equally desirable to an individual. These changes require an expansion in the notion of stability, with the existence of stable matches with preference lists which have ties having been proven by Irving [5]. Moving beyond preference lists, another real-world obstacle to labor and residency markets is the existence of family and spousal connections. While strategy-proofness and stability may be important theoretical concerns, a practical matter that hospitals have to contend with in the residency match is preventing couples from being assigned to residency programs far from each other. When couples were modeled as having preferences over pairs of hospitals, it was shown that the set of stable matches could be empty [14]. However, a generalized applicant-proposing deferred acceptance algorithm was proposed by Roth and Peranson which takes multiple passes to produce a stable matching with high probability [16]. These constructions are more general and applicable than our simple symmetric market and, if work by Pittel [13] is an indicator, they also lead to deeper, more complex relations that build upon the ones proven here.

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